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String hypothesis for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ mirror

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Abstract: We discuss the states which contribute in the thermodynamic limit of the mirror theory, the latter is obtained from the light-cone gauge-fixed string theory in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background by the double-Wick rotation. We analyze the Bethe-Yang equations for the mirror theory and formulate the string hypothesis. We show that in the thermodynamic limit solutions of the Bethe-Yang equations arrange themselves into Bethe string configurations similar to the ones appearing in the Hubbard model. We also derive a set of equations describing the bound states and the Bethe string configurations of the mirror theory.

Keywords: AdS-CFT Correspondence, Bethe Ansatz

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Contents

1 Introduction

The AdS/CFT correspondence [1] offers new profound insights into a strong coupling dynamics of gauge theories. In the basic case of the duality between type IIB superstrings on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM one may even hope to find an exact solution of the tree-level string theory, and, therefore, to solve the dual gauge theory in the 't Hooft limit. This would be done by employing the conjectured quantum integrability of the $AdS_5 \times S^5$ superstring which is supported by classical integrability [2] of the Green-Schwarz action [3], and by one-loop integrability of the dual gauge theory [4, 5].

Solving string theory is a multi-step problem. One starts by imposing the light-cone gauge for the $AdS_5 \times S^5$ superstring, and obtains a 2-d non-linear sigma model defined on a cylinder of circumference equal to the light-cone momentum P_+ [6, 7]. The gauge-fixed Hamiltonian is equal to $E - J$ and, therefore, its spectrum determines the spectrum of scaling dimensions of gauge theory operators. To find the spectrum, one first takes the decompactification limit [8]–[11], i.e. the limit where P_+ goes to infinity, while keeping the string tension g fixed. Then, one is left with a world-sheet theory on a plane which has a massive spectrum and well-defined asymptotic states (particles). This reduces the spectral problem to finding dispersion relations for particles and the S-matrices describing their pairwise scattering. Quantum integrability then implies factorization of multi-particle scattering into a sequence of two-body events [12].

To define the S-matrix, one should deal with particles with arbitrary world-sheet momenta which requires to give up the level-matching condition. As a result, the manifest

$\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2) \subset \mathfrak{psu}(2,2|4)$ symmetry algebra of the light-cone string theory gets enhanced by two central charges [13]. The same centrally-extended symmetry algebra also appears in the dual gauge theory [14].

An important observation made in [14] is that the dispersion relation for fundamental particles is uniquely determined by the symmetry algebra of the model. Moreover, the matrix structure of their S-matrix is uniquely fixed by the algebra, the Yang-Baxter equation and the generalized physical unitarity condition [14–16].

The S-matrix is thus determined up to an overall scalar function $\sigma(p_1, p_2)$ — the so-called dressing factor [17]. Its functional form was conjectured in [17] by discretizing the integral equations [18] describing classical spinning strings [19, 20], and using insights from gauge theory [21]. It was proposed in [22] that the dressing factor satisfies a crossing equation. Combining the functional form of the dressing factor together with the first two known orders in the strong coupling expansion [17, 23], a set of solutions to the crossing equation in terms of an all-order strong coupling asymptotic series has been proposed [24]. Opposite to the strong coupling expansion, gauge theory perturbative expansion of the dressing factor is in powers of g and it has a finite radius of convergence. An interesting proposal for the exact dressing factor has been put forward in [25], and passed many tests [26]–[29]. Thus, one can adopt the working assumption that the exact dressing factor and, therefore, the S-matrix are established.

Having found the exact dispersion relation and the S-matrix, the next step is to determine bound states of the model. Analysis reveals that all bound states are those of elementary particles [30], and comprise into the tensor product of two $4Q$-dim atypical totally symmetric multiplets of the centrally-extended symmetry algebra $\mathfrak{su}(2|2)$ [31].

Having understood the spectrum of the light-cone string sigma model on a plane, one has to "upgrade" the findings to a cylinder. All physical string configurations (and dual gauge theory operators) are characterized by a finite value of P_+, and as such they are excitations of a theory on a cylinder. The first step in determining the finite-size spectrum is to impose the periodicity condition on the Bethe wave function. This leads to a system of equations on the particle momenta known as the Bethe-Yang equations. In the AdS/CFT context these equations are usually referred to as the asymptotic Bethe ansatz [32, 33]. The $\mathrm{AdS}_5 \times \mathrm{S}^5$ string S-matrix has a complicated matrix structure which results at the end in a set of nested Bethe equations [14, 34, 35].

The Bethe-Yang equations determine any power-like $1/P_+$ corrections to energy of multi-particle states. To find the exact spectrum for finite values of string tension and P_+, one may try to generalize the thermodynamic Bethe ansatz (TBA), originally developed for relativistic integrable models [36], to the light-cone string theory.

The TBA approach might allow one to relate the exact string spectrum to proper thermodynamic quantities of the mirror theory obtained from the light-cone string model by means of a double-Wick rotation. The mirror theory lives on a plane at temperature $1/P_+$, and, in particular, its Gibbs free energy is equal to the density of the ground state energy of the string model. It should be also possible to find the energies of excited states by analytic continuation of the TBA equations, see e.g [39]–[43] for some relativistic examples.

Since the light-cone string model is not Lorentz-invariant, the mirror theory is governed by a different Hamiltonian and therefore has very different dynamics. Thus, to implement the TBA approach one has to study the mirror theory in detail. The first step in this direction has been taken in [16], where the Bethe-Yang equations for fundamental particles of the mirror model were derived. Another result of [16] was the classification of mirror bound states according to which they comprise into the tensor product of two $4Q$-dim atypical totally anti-symmetric multiplets of the centrally-extended algebra $\mathfrak{su}(2|2)$. This observation was used in the derivation [37] of the four-loop scaling dimension of the Konishi operator by means of Lüscher's formulae [38]. We consider this derivation as prime evidence for the validity of the mirror theory approach.

In this paper we take the next step in studying the mirror theory, and identify the states that contribute in the thermodynamic limit. We use the Bethe-Yang equations of [16] and the fusion procedure, see e.g. [44], to write down the equations for the complete spectrum of the mirror theory. We use the observation of [31] that the equations for auxiliary roots can be interpreted as the Lieb-Wu equations for an inhomogeneous Hubbard model [45], and notice that the inhomogeneous Hubbard model becomes homogeneous in the limit of the infinite real momenta of the mirror particles. This observation allows us to formulate the string hypothesis for the mirror theory. We show that the solutions of the Bethe-Yang equations in the thermodynamic limit arrange themselves into Bethe string configurations similar to the ones appearing in the Hubbard model [46]. We then derive a set of equations describing the bound states of the mirror theory and the Bethe string configurations. These equations can be readily used to derive a set of TBA equations for the free energy of the mirror model following a textbook route, see e.g. [46]. The resulting equations are however complicated and we postpone their discussion for future publication.

1.1 Bethe-Yang equations

The Bethe-Yang equations for fundamental particles and bound states of the mirror theory defined on a circle of large circumference R are derived by using the $\mathfrak{su}(2|2)\oplus\mathfrak{su}(2|2)$-invariant S-matrix [16] and the fusion procedure, and are of the form

$$\begin{aligned} 1 &= e^{i\widetilde{p}_k R}\prod_{\substack{l=1\\ l\neq k}}^{K^{\rm I}} S_{\mathfrak{sl}(2)}^{Q_kQ_l}(x_k,x_l)\prod_{\alpha=1}^{2}\prod_{l=1}^{K^{\rm II}_{(\alpha)}}\frac{x_k^- - y_l^{(\alpha)}}{x_k^+ - y_l^{(\alpha)}}\sqrt{\frac{x_k^+}{x_k^-}} \\ -1 &= \prod_{l=1}^{K^{\rm I}}\frac{y_k^{(\alpha)}-x_l^-}{y_k^{(\alpha)}-x_l^+}\sqrt{\frac{x_l^+}{x_l^-}}\prod_{l=1}^{K^{\rm III}_{(\alpha)}}\frac{v_k^{(\alpha)}-w_l^{(\alpha)}-\frac{i}{g}}{v_k^{(\alpha)}-w_l^{(\alpha)}+\frac{i}{g}} \\ 1 &= \prod_{l=1}^{K^{\rm II}_{(\alpha)}}\frac{w_k^{(\alpha)}-v_l^{(\alpha)}+\frac{i}{g}}{w_k^{(\alpha)}-v_l^{(\alpha)}-\frac{i}{g}}\prod_{\substack{l=1\\ l\neq k}}^{K^{\rm III}_{(\alpha)}}\frac{w_k^{(\alpha)}-w_l^{(\alpha)}-\frac{2i}{g}}{w_k^{(\alpha)}-w_l^{(\alpha)}+\frac{2i}{g}}. \end{aligned} \tag{1.1}$$

Here $\widetilde{p}_k$ is the real momentum of a physical mirror particle which can be either a fundamental particle or a Q-particle bound state. We will often refer to such a particle as a Q-particle, a 1-particle being a fundamental one. Then, $K^{\rm I}$ is the number of Q-particles, and $K^{\rm II}_{(\alpha)}$ and $K^{\rm III}_{(\alpha)}$ are the numbers of auxiliary roots $y_k^{(\alpha)}$ and $w_k^{(\alpha)}$ of the second and third levels of the nested Bethe ansatz, and $\alpha = 1, 2$ because the scattering matrix is the tensor product of the two $\mathfrak{su}(2|2)$-invariant S-matrices. We will often refer to K's as to excitation numbers. The parameters v are related to y as $v = y + \frac{1}{y}$. The parameters $x^{\pm}$ are functions of the string tension g, the momentum $\widetilde{p}$ and the number of constituents Q of a Q-particle, and their explicit expressions can be found in appendix A, eq. (A.6).

The function $S^{Q_k Q_l}_{\mathfrak{sl}(2)}(x_k, x_l)$ is the two-particle scalar S-matrix which describes the scattering of a Q_k-particle with momentum $\widetilde{p}_k$ and a Q_l-particle with momentum $\widetilde{p}_l$ in the $\mathfrak{sl}(2)$ sector of the mirror theory. The S-matrix can be found by using the fusion procedure and the following $\mathfrak{sl}(2)$ S-matrix of the fundamental particles

$$S^{11}_{\mathfrak{sl}(2)}(x_1, x_2) = \sigma_{12}^{-2} s_{12}\,, \quad s_{12} = \frac{x_1^+ - x_2^-}{x_1^- - x_2^+} \frac{1 - \frac{1}{x_1^- x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}}\,, \tag{1.2}$$

where σ_{12} is the dressing factor [17] that depends on $x^{\pm}$ and g. Its exact form was conjectured in [25] but we will not need it here. For complex values of the momenta $\widetilde{p}_1$, $\widetilde{p}_2$ the S-matrix (1.2) exhibits a pole at $x_1^- = x_2^+$, and it is this pole that leads to the existence of a Q-particle bound state satisfying the bound state equation [16]

$$x_1^- = x_2^+\,, \quad x_2^- = x_3^+\,, \ \ldots, x_{Q-1}^- = x_Q^+\,. \tag{1.3}$$

The equation has a unique solution in the physical region of the mirror theory defined by $\operatorname{Im} x^{\pm} < 0$ [16], and it is used in the fusion procedure. It implies that the S-matrix $S^{Q_k Q_l}_{\mathfrak{sl}(2)}(x_k, x_l)$ depends only on the total real momenta of the Q-particles.

Since $S^{11}_{\mathfrak{sl}(2)}(x_k, x_l)$ can be also written as

$$S^{11}_{\mathfrak{sl}(2)}(x_1, x_2) = \frac{u_1 - u_2 + \frac{2i}{g}}{u_1 - u_2 - \frac{2i}{g}} \times \left(\frac{1 - \frac{1}{x_1^+ x_2^-}}{1 - \frac{1}{x_1^- x_2^+}} \sigma_{12} \right)^{-2}\,, \tag{1.4}$$

the Q-particle bound state equations (1.3) can be cast in the form

$$u_j - u_{j+1} - \frac{2i}{g} = 0 \iff x_j^- = x_{j+1}^+\,, \quad j = 1, 2, \ldots, Q-1\,. \tag{1.5}$$

Then, the solution to (1.5) is simply given by the Bethe string

$$u_j = u + (Q + 1 - 2j)\frac{i}{g}\,, \quad j = 1, \ldots, Q\,, \quad u \in \mathbf{R}\,, \tag{1.6}$$

where the real rapidity u determines the momentum of the bound state through eq. (A.13) from appendix A.

By taking the complex conjugate of the first Bethe-Yang equation in (1.1) one can easily see that the unitarity of the S-matrix (1.2) implies that for real values of $\widetilde{p}_k$ the auxiliary roots y either come in pairs $y_2 = 1/y_1^*$ or lie on unit circle. As a consequence the variables v and w come in complex conjugate pairs, or are real.

It is the set of Bethe-Yang equations (1.1) we will be using in the paper to analyze the solutions which contribute in the thermodynamic limit. However, before starting the analysis we would like to show the relation of the last two equations in (1.1) for the auxiliary roots to the Lieb-Wu equations for the Hubbard model.

1.2 Relation to the Lieb-Wu equations

Let us recall that the Lieb-Wu equations are the Bethe equations for the Hubbard model and have the form [45, 46]

$$e^{-i\phi}e^{iq_kL} = \prod_{l=1}^{M} \frac{\lambda_l - \sin q_k - i\frac{U}{4}}{\lambda_l - \sin q_k + i\frac{U}{4}}\,, \tag{1.7}$$
$$\prod_{l=1}^{N} \frac{\lambda_k - \sin q_l - i\frac{U}{4}}{\lambda_k - \sin q_l + i\frac{U}{4}} = \prod_{\substack{l=1\\ l\neq l}}^{M} \frac{\lambda_k - \lambda_l - i\frac{U}{2}}{\lambda_k - \lambda_l + i\frac{U}{2}}\,,$$

where U is the coupling constant of the Hubbard model, q_k, $k = 1,\dots,N$, and λ_l, $l = 1,\dots,M$ are charge momenta and spin rapidities, respectively. The arbitrary constant ϕ is a twist which has the physical interpretation of the magnetic flux.

To relate the Bethe-Yang equations (1.1) for the auxiliary roots to the Lieb-Wu equations[1] let us make the following change

$$y = ie^{-iq}\,, \quad v = 2\sin q\,, \quad w = 2\lambda\,.$$

Then the second and third equations in (1.1) can be cast in the form

$$-\prod_{l=1}^{K^{\mathrm{I}}} \frac{e^{-iq_k^{(\alpha)}} + ix_l^-}{e^{-iq_k^{(\alpha)}} + ix_l^+}\sqrt{\frac{x_l^+}{x_l^-}} = \prod_{l=1}^{K^{\mathrm{III}}_{(\alpha)}} \frac{\sin q_k^{(\alpha)} - \lambda_l^{(\alpha)} + \frac{i}{2g}}{\sin q_k^{(\alpha)} - \lambda_l^{(\alpha)} - \frac{i}{2g}}\,,$$
$$\prod_{l=1}^{K^{\mathrm{II}}_{(\alpha)}} \frac{\lambda_k^{(\alpha)} - \sin q_l^{(\alpha)} - \frac{i}{2g}}{\lambda_k^{(\alpha)} - \sin q_l^{(\alpha)} + \frac{i}{2g}} = \prod_{\substack{l=1\\ l\neq k}}^{K^{\mathrm{III}}_{(\alpha)}} \frac{\lambda_k^{(\alpha)} - \lambda_l^{(\alpha)} - \frac{i}{g}}{\lambda_k^{(\alpha)} - \lambda_l^{(\alpha)} + \frac{i}{g}}\,.$$

Thus, we see that if for each value of α we identify $g \to 2/U$, $K^{\mathrm{I}} \to L$, $K^{\mathrm{III}}_{(\alpha)} \to M$ and $K^{\mathrm{II}}_{(\alpha)} \to N$, then we get two copies of equations which can be interpreted as the Bethe equations for an inhomogeneous Hubbard model. The inhomogeneities are determined by the real momenta of the physical particles of the mirror theory. One can easily see that in the limit $\widetilde{p} \to \infty$ the parameters $x^\pm$ behave as $x^+ \to 0$, $x^- \to \infty$ and one obtaines the homogeneous Lieb-Wu equations (1.7) with $\phi = (L-2)\pi/2$.

The relation to the Hubbard model leads us to a natural conjecture that in the thermodynamic limit where $K^{\mathrm{I}}, K^{\mathrm{II}}_{(\alpha)}, K^{\mathrm{III}}_{(\alpha)} \to \infty$ the auxiliary roots y and w will arrange themselves in vw- and w-strings that in the case of the Hubbard model are called the k-Λ and Λ strings [46].

2 String hypothesis

In this section we argue that in the thermodynamic limit $R, K^{\mathrm{I}}, K^{\mathrm{II}}_{(\alpha)}, K^{\mathrm{III}}_{(\alpha)} \to \infty$ with K^{I}/R and so on fixed the solutions of the Bethe-Yang equations (1.1) are composed of the following four different classes of Bethe strings

[1]This relation was first observed in [31] and is quite natural taking into account that the $\mathfrak{su}(2|2)$-invariant S-matrix coincides with Shastry's R-matrix [49] up to a scalar factor [31, 34].

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1. A single Q-particle with real momentum $\widetilde{p}_k$ or, equivalently, rapidity u_k
2. A single $y^{(\alpha)}$-particle corresponding to an auxiliary root $y^{(\alpha)}$ with $|y^{(\alpha)}| = 1$
3. $2M$ roots $y^{(\alpha)}$ and M roots $w^{(\alpha)}$ combining into a single $M|vw^{(\alpha)}$-string

$$\begin{aligned} v_j^{(\alpha)} &= v^{(\alpha)} + (M+2-2j)\frac{i}{g}\,, \quad v_{-j}^{(\alpha)} = v^{(\alpha)} - (M+2-2j)\frac{i}{g}\,, \quad j = 1,\dots,M\,, \\ w_j^{(\alpha)} &= v^{(\alpha)} + (M+1-2j)\frac{i}{g}\,, \qquad j = 1,\dots,M\,, \qquad v \in \mathbf{R}\,. \end{aligned} \tag{2.1}$$

4. N roots $w^{(\alpha)}$ combining into a single $N|w^{(\alpha)}$-string

$$w_j^{(\alpha)} = w^{(\alpha)} + \frac{i}{g}(N+1-2j)\,, \quad j = 1,\dots,N\,, \quad w \in \mathbf{R}\,. \tag{2.2}$$

This includes $N = 1$ which has a single real root $w^{(\alpha)}$.

According to the string hypothesis for large R almost all solutions of the Bethe-Yang equations (1.1) are approximately given by these Bethe strings with corrections decreasing exponentially in R. The last three types are in fact the same as in the Hubbard model [46]. Every solution of (1.1) corresponds to a particular configuration of the Bethe strings, and consists of

1. N_Q Q-particles, $Q = 1, 2, \dots, \infty$
2. $N_y^{(\alpha)}$ $y^{(\alpha)}$-particles
3. $N_{M|vw}^{(\alpha)}$ $M|vw^{(\alpha)}$-strings, $\alpha = 1, 2$; $M = 1, 2, \dots, \infty$
4. $N_{N|w}^{(\alpha)}$ $N|w^{(\alpha)}$-strings, $\alpha = 1, 2$; $N = 1, 2, \dots, \infty$

We have infinitely many states of all these kinds in the thermodynamic limit. The numbers N_Q, $N_y^{(\alpha)}$, $N_{M|vw}^{(\alpha)}$, $N_{N|w}^{(\alpha)}$ are called the occupation numbers of the root configuration under consideration, and they obey the 'sum rules'

$$\begin{aligned} K^{\mathrm{I}} &= \sum_{Q=1}^{\infty} N_Q\,, \\ K_{(\alpha)}^{\mathrm{II}} &= N_y^{(\alpha)} + \sum_{M=1}^{\infty} 2M\, N_{M|vw}^{(\alpha)}\,, \\ K_{(\alpha)}^{\mathrm{III}} &= \sum_{M=1}^{\infty} M\left(N_{M|vw}^{(\alpha)} + N_{M|w}^{(\alpha)}\right)\,. \end{aligned} \tag{2.3}$$

Solutions of the Bethe-Yang equations (1.1) with no coinciding roots, and having excitation numbers satisfying the following inequalities

$$\sum_{Q=1}^{\infty} Q\, N_Q \equiv K_{\mathrm{tot}}^{\mathrm{I}} \geq K_{(\alpha)}^{\mathrm{II}} \geq 2K_{(\alpha)}^{\mathrm{III}} \tag{2.4}$$

are called regular. Solutions which differ by ordering of roots are considered as equivalent.

We expect in analogy with the Hubbard model that each regular solution corresponds to a highest weight state of the four $\mathfrak{su}(2)$ subalgebras of the $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ symmetry algebra of the model and vise versa. The Dynkin labels are related to the excitation numbers as follows

$$s_\alpha = K^{\rm I}_{\rm tot} - K^{\rm II}_{(\alpha)}\,, \quad q_\alpha = K^{\rm II}_{(\alpha)} - 2K^{\rm III}_{(\alpha)}\,.$$

This follows from the fact that a Q-particle is a bound state of Q fundamental particles.

In the remaining part of the section we explain how the Bethe string configurations can be found.

2.1 $M|vw$-strings

Let us recall that to find the Q-particle bound states one should consider complex values of particle's momenta and take the limit $R \to \infty$ keeping the numbers $K^{\rm I}$, $K^{\rm II}_{(\alpha)}$ and $K^{\rm III}_{(\alpha)}$ of the physical particles and auxiliary roots finite. The Bethe string configurations of the auxiliary roots can be also found in a similar way.

To determine the string configurations of $y_k^{(\alpha)}$ roots we assume that the momenta of physical particles are real, and take $K^{\rm I}$ to infinity keeping $K^{\rm II}_{(\alpha)}$ and $K^{\rm III}_{(\alpha)}$ finite.

Then, one can easily show that

$$\prod_{l=1}^{K^{\rm I}} \frac{y_k^{(\alpha)} - x_l^+}{y_k^{(\alpha)} - x_l^-}\sqrt{\frac{x_l^-}{x_l^+}} \;\longrightarrow\; 0 \;\; \text{if} \;\; |y_k^{(\alpha)}| < 1\,,$$

and

$$\prod_{l=1}^{K^{\rm I}} \frac{y_k^{(\alpha)} - x_l^+}{y_k^{(\alpha)} - x_l^-}\sqrt{\frac{x_l^-}{x_l^+}} \;\longrightarrow\; \infty \;\; \text{if} \;\; |y_k^{(\alpha)}| > 1\,.$$

If $|y_k^{(\alpha)}| = 1$ then the absolute value of the product is equal to 1.

We can consider roots with $\alpha = 1$, denote them as y_k, v_k and w_k, and assume without loss of generality that $|y_1| < 1$. Then the Bethe-Yang equation for y_1 in (1.1) takes the form

$$-1 = \prod_{l=1}^{K^{\rm I}} \frac{y_1 - x_l^+}{y_1 - x_l^-}\sqrt{\frac{x_l^-}{x_l^+}} \prod_{l=1}^{K^{\rm III}} \frac{v_1 - w_l + \frac{i}{g}}{v_1 - w_l - \frac{i}{g}} \;\longrightarrow\; -1 = 0 \times \prod_{l=1}^{K^{\rm III}} \frac{v_1 - w_l + \frac{i}{g}}{v_1 - w_l - \frac{i}{g}}\,. \tag{2.5}$$

Thus, to satisfy this equation we must have a root w_1 such that

$$v_1 - w_1 - \frac{i}{g} = 0 \implies v_1 = w_1 + \frac{i}{g}\,, \tag{2.6}$$

and computing y_1 by using v_1 we should keep the solution with $|y_1| < 1$.

The equation for w_1 takes the form

$$1 = \prod_{l=1}^{K^{\rm II}} \frac{w_1 - v_l - \frac{i}{g}}{w_1 - v_l + \frac{i}{g}} \prod_{l=2}^{K^{\rm III}} \frac{w_1 - w_l + \frac{2i}{g}}{w_1 - w_l - \frac{2i}{g}} \;\longrightarrow\; 1 = \frac{1}{0} \times \prod_{l=2}^{K^{\rm II}} \frac{w_1 - v_l - \frac{i}{g}}{w_1 - v_l + \frac{i}{g}} \prod_{l=2}^{K^{\rm III}} \frac{w_1 - w_l + \frac{2i}{g}}{w_1 - w_l - \frac{2i}{g}}\,.$$

We have to assume that there is a root v_2 such that

$$w_1 - v_2 - \frac{i}{g} = 0 \implies v_2 = w_1 - \frac{i}{g} \,. \tag{2.7}$$

Otherwise if there is no such v_2 then $w_1 - w_2 + 2i/g = 0$, and, therefore from (2.6), $v_1 - w_2 + i/g = 0$, and we get into a contradiction with (2.5).

Then the Bethe-Yang equation for y_2 in (1.1) acquires the form

$$-1 = \prod_{l=1}^{K^{\rm I}} \frac{y_2 - x_l^+}{y_2 - x_l^-} \sqrt{\frac{x_l^-}{x_l^+}} \prod_{l=1}^{K^{\rm III}} \frac{v_2 - w_l + \frac{i}{g}}{v_2 - w_l - \frac{i}{g}} \quad \longrightarrow \quad -1 = 0 \times \prod_{l=1}^{K^{\rm I}} \frac{y_2 - x_l^+}{y_2 - x_l^-} \sqrt{\frac{x_l^-}{x_l^+}} \prod_{l=2}^{K^{\rm III}} \frac{v_2 - w_l + \frac{i}{g}}{v_2 - w_l - \frac{i}{g}} \,.$$

Now, if we take y_2 with $|y_2| > 1$, then we can satisfy this equation and obtain a $1|vw$-string

$$v_1 = v + \frac{i}{g} \,, \;\; |y_1| < 1 \,, \quad v_2 = v - \frac{i}{g} \,, \;\; |y_2| > 1 \,, \quad w_1 = v \,, \quad v \in \mathbf{R} \,,$$

where the roots y_i satisfy $y_2 = 1/y_1^*$.

On the other hand, if we take y_2 with $|y_2| \le 1$, then we get the same conditions we had for y_1 in (2.5), and, therefore, there should exist a root w_2 such that

$$v_2 - w_2 - \frac{i}{g} = 0 \implies w_2 = v_2 - \frac{i}{g} = w_1 - \frac{2i}{g} \,.$$

If we stop here we get a $2|vw$-string with

$$\begin{array}{llll} w_1 = v + \frac{i}{g} \,, & w_2 = v - \frac{i}{g} \,, & v \in \mathbf{R} \,, & \\ v_1 = v + \frac{2i}{g} \,, & |y_1| < 1 \,, & v_{-1} = v - \frac{2i}{g} \,, & y_{-1} = \frac{1}{y_1^*} \,, \\ v_2 = v \,, & |y_2| \le 1 \,, & v_{-2} = v \,, & y_{-2} = \frac{1}{y_2} \,, \end{array}$$

where we denoted $y_4 \equiv y_{-1}$ and $y_3 \equiv y_{-2}$.

If we continue the process we get a general $M|vw$-string characterized by the following set of equations

$$\begin{array}{lll} w_j = v + (M + 1 - 2j)\frac{i}{g} \,, & j = 1, \ldots, M \,, & v \in \mathbf{R} \,, \\ v_j = v + (M + 2 - 2j)\frac{i}{g} \,, & v_{-j} = v - (M + 2 - 2j)\frac{i}{g} \,, & j = 1, \ldots, M \,, \end{array} \tag{2.8}$$

where the corresponding roots y_j and y_{-j} are related as $y_{-j} y_j^* = 1$ if $j \neq \frac{M+2}{2}$, and $y_{\frac{M+2}{2}} \, y_{-\frac{M+2}{2}} = 1$ (that may happen only for even M). Computing them by using v_j and v_{-j} we should keep the solutions with $|y_j| \le 1$ and $|y_{-j}| \ge 1$ for $1 \le j \le M$. It is worth mentioning that even though $v_j = v_{-M-2+j}$ for $j = 2, 3, \ldots, M$, this requirement guarantees that all the roots y_j in the string are different. In particular, for $j = 2, 3, \ldots, M$ the roots y_j and y_{-M-2+j} are related to each other as $y_j y_{-M-2+j} = 1$. It is also interesting that $\mathrm{Im}(y_1) < 0$ for any string. This is the condition the parameters $x^{\pm}$ have to satisfy because it defines the physical region of the mirror theory. In general, however, the y-roots can take arbitrary values.

2.2 $M|w$-strings

As we discussed in the previous subsection if we have a root y with $|y| < 1$ (or $|y| > 1$) then in the thermodynamic limit we unavoidably get a $M|vw$-string. So, we just need to consider the case where $|y_1| = 1$ that is v_1 is real, and takes the values $-2 < v_1 < 2$. Then, taking the limit $K^{\rm II}_{(\alpha)} \to \infty$ and keeping $K^{\rm III}_{(\alpha)}$ finite, one can easily see that

$$\prod_{l=1}^{K^{\rm II}_{(\alpha)}} \frac{w_k^{(\alpha)} - v_l^{(\alpha)} - \frac{i}{g}}{w_k^{(\alpha)} - v_l^{(\alpha)} + \frac{i}{g}} \longrightarrow 0 \;\; \text{if} \;\; {\rm Im}(w_k^{(\alpha)}) > 0\,,$$

and

$$\prod_{l=1}^{K^{\rm II}_{(\alpha)}} \frac{w_k^{(\alpha)} - v_l^{(\alpha)} - \frac{i}{g}}{w_k^{(\alpha)} - v_l^{(\alpha)} + \frac{i}{g}} \longrightarrow \infty \;\; \text{if} \;\; {\rm Im}(w_k^{(\alpha)}) < 0\,,$$

Thus, assuming for definiteness that ${\rm Im}(w_1) > 0$, we get that the first factor in the third equation in (1.1) is exponentially decreasing, and therefore we should have

$$w_2 = w_1 - \frac{2i}{g}\,.$$

Then there are two cases. First we could have

$${\rm Im}(w_2) < 0\,,$$

and one can easily check that the equation for w_2 is satisfied. The reality condition would also give $w_2 = w_1^*$, and one gets a $2|w$-string

$$w_1 = w + \frac{i}{g}\,, \quad w_2 = w - \frac{i}{g}\,, \quad w \in \mathbf{R}\,.$$

If ${\rm Im}(w_2) > 0$ then there is $w_3 = w_2 - \frac{2i}{g}$, and the procedure repeats itself. As a result we get a $M|w$-string

$$w_j = w + \frac{i}{g}(M - 2j + 1)\,, \quad j = 1, \dots, M\,, \quad w \in \mathbf{R}\,,$$

which is the usual Bethe string.

3 Bethe-Yang equations for string configurations

Next we express the Bethe-Yang equations (1.1) in terms of real physical momenta $\widetilde{p}$ of Q-particles, auxiliary momenta q of y-particles with $y = ie^{-iq}$, real coordinates v of centers of vw-strings, and real coordinates w of centers of w-strings.

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3.1 Bethe-Yang equations for Q-particles

The first step is to rewrite the first equation in (1.1) in terms of momenta $q_k^{(\alpha)}$, $k = 1, \dots, N_y^{(\alpha)}$ of $y^{(\alpha)}$-particles, and coordinates $v_{k,M}^{(\alpha)}$, $k = 1, \dots, N_{M|vw}^{(\alpha)}$ of vw-strings. A simple computation gives

$$1 = e^{i\widetilde{p}_k R} \prod_{\substack{l=1 \\ l \neq k}}^{K^{\mathrm{I}}} S_{\mathfrak{sl}(2)}^{Q_k Q_l}(x_k, x_l) \prod_{\alpha=1}^{2} \prod_{l=1}^{N_y^{(\alpha)}} \frac{x_k^- - y_l^{(\alpha)}}{x_k^+ - y_l^{(\alpha)}} \sqrt{\frac{x_k^+}{x_k^-}} \prod_{M=1}^{\infty} \prod_{l=1}^{N_{M|vw}^{(\alpha)}} S_{xv}^{Q_k M}(x_k, v_{l,M}^{(\alpha)}) , \tag{3.1}$$

Here the auxiliary S-matrix is given by

$$S_{xv}^{Q_k M}(x_k, v_{l,M}^{(\alpha)}) = \frac{x_k^- - y_{l,M}^{(\alpha)+}}{x_k^+ - y_{l,M}^{(\alpha)+}} \frac{x_k^- - y_{l,M}^{(\alpha)-}}{x_k^+ - y_{l,M}^{(\alpha)-}} \frac{x_k^+}{x_k^-} \prod_{j=1}^{M-1} \frac{u_k^- - v_{l,M}^{(\alpha)-} - \frac{2i}{g} j}{u_k^+ - v_{l,M}^{(\alpha)+} + \frac{2i}{g} j} , \tag{3.2}$$

where

$$y_{l,M}^{(\alpha)\pm} = x(v_{l,M}^{(\alpha)\pm}) , \quad v_{l,M}^{(\alpha)\pm} = v_{l,M}^{(\alpha)} \pm \frac{i}{g} M ,$$

and $x(u)$ is defined in (A.15).

For what follows it is convenient to adopt the following notation

$$\begin{aligned} &N_y = N_y^{(1)} + N_y^{(2)} , \qquad y_l = y_l^{(1)} , \qquad l = 1, \dots, N_y^{(1)} , \quad y_{N_y^{(1)}+l} = y_l^{(2)} , \quad l = 1, \dots, N_y^{(2)} , \\ &v_{k,M}^{(1)} = v_{k,M} , \qquad v_{k,M}^{(2)} = v_{k,-M} . \end{aligned} \tag{3.3}$$

With this notation the Bethe-Yang equations (3.1) for Q-particles take a slightly simpler form

$$1 = e^{i\widetilde{p}_k R} \prod_{\substack{l=1 \\ l \neq k}}^{K^{\mathrm{I}}} S_{\mathfrak{sl}(2)}^{Q_k Q_l}(x_k, x_l) \prod_{l=1}^{N_y} \frac{x_k^- - y_l}{x_k^+ - y_l} \sqrt{\frac{x_k^+}{x_k^-}} \prod_{\substack{M=-\infty \\ M \neq 0}}^{\infty} \prod_{l=1}^{N_{M|vw}} S_{xv}^{Q_k M}(x_k, v_{l,M}) , \tag{3.4}$$

where the auxiliary S-matrix is given by (3.2) with

$$y_{l,M}^{\pm} = x(v_{l,M}^{\pm}) , \quad v_{l,M}^{\pm} = v_{l,M} \pm \frac{i}{g} |M| . \tag{3.5}$$

3.2 Bethe-Yang equations for y-particles

Next we take a $y^{(\alpha)}$-particle with the root $y_k^{(\alpha)} = ie^{-iq_k^{(\alpha)}}$ and rewrite the second equation in (1.1) in terms of coordinates $v_{k,M}^{(\alpha)}$, $k = 1, \dots, N_{M|vw}^{(\alpha)}$ of vw-strings, and coordinates $w_{k,N}^{(\alpha)}$, $k = 1, \dots, N_{N|w}^{(\alpha)}$ of w-strings. The result is

$$-1 = \prod_{l=1}^{K^{\mathrm{I}}} \frac{y_k^{(\alpha)} - x_l^-}{y_k^{(\alpha)} - x_l^+} \sqrt{\frac{x_l^+}{x_l^-}} \prod_{M=1}^{\infty} \prod_{l=1}^{N_{M|vw}^{(\alpha)}} \frac{v_k^{(\alpha)} - v_{l,M}^{(\alpha)+}}{v_k^{(\alpha)} - v_{l,M}^{(\alpha)-}} \prod_{N=1}^{\infty} \prod_{l=1}^{N_{N|w}^{(\alpha)}} \frac{v_k^{(\alpha)} - w_{l,N}^{(\alpha)+}}{v_k^{(\alpha)} - w_{l,N}^{(\alpha)-}} , \tag{3.6}$$

where

$$w_{l,N}^{(\alpha)\pm} = w_{l,N}^{(\alpha)} \pm \frac{i}{g} N . \tag{3.7}$$

In fact we get the same equation for any root $y_k^{(\alpha)}$, no matter if it is a root of a y-particle or a vw-string.

3.3 Bethe-Yang equations for w-strings

Now we take a $K|w$-string with the coordinates $w_{k,K}^{(\alpha)}$. The last equations in (1.1) can be written in the form

$$-1 = \prod_{l=1}^{N_y^{(\alpha)}} \frac{w_k^{(\alpha)} - v_l^{(\alpha)} + \frac{i}{g}}{w_k^{(\alpha)} - v_l^{(\alpha)} - \frac{i}{g}} \prod_{N=1}^{\infty} \prod_{l=1}^{N_{N|w}^{(\alpha)}} \frac{w_k^{(\alpha)} - w_{l,N}^{(\alpha)+} + \frac{i}{g}}{w_k^{(\alpha)} - w_{l,N}^{(\alpha)-} + \frac{i}{g}} \frac{w_k^{(\alpha)} - w_{l,N}^{(\alpha)+} - \frac{i}{g}}{w_k^{(\alpha)} - w_{l,N}^{(\alpha)-} - \frac{i}{g}} . \tag{3.8}$$

It is interesting that the equation has no dependence on the coordinates of vw-strings. Multiplying K equations in (3.8) with the roots $w_k^{(\alpha)}$ that form the $K|w$-string, we get

$$(-1)^K = \prod_{l=1}^{N_y^{(\alpha)}} \frac{w_{k,K}^{(\alpha)+} - v_l^{(\alpha)}}{w_{k,K}^{(\alpha)-} - v_l^{(\alpha)}} \prod_{N=1}^{\infty} \prod_{l=1}^{N_{N|w}^{(\alpha)}} S_{vv}^{KN}(w_{k,K}^{(\alpha)}, w_{l,N}^{(\alpha)}) , \tag{3.9}$$

where the auxiliary S-matrix is

$$S_{vv}^{KM}(u,u') = \frac{u-u'-\frac{i}{g}(K+M)}{u-u'+\frac{i}{g}(K+M)} \frac{u-u'-\frac{i}{g}(M-K)}{u-u'+\frac{i}{g}(M-K)} \times \prod_{j=1}^{K-1} \left(\frac{u-u'-\frac{i}{g}(M-K+2j)}{u-u'+\frac{i}{g}(M-K+2j)} \right)^2 . \tag{3.10}$$

3.4 Bethe-Yang equations for vw-strings

Finally we take a $K|vw$-string with the coordinates $v_{k,K}^{(\alpha)}$, and multiply $2K$ equations (3.6) with the roots $y_k^{(\alpha)}$ that form the $K|vw$-string. The resulting equation takes the form

$$1 = \prod_{l=1}^{K^{\mathrm{I}}} S_{xv}^{Q_l K}(x_l, v_{k,K}^{(\alpha)}) \prod_{M=1}^{\infty} \prod_{l=1}^{N_{M|vw}^{(\alpha)}} S_{vv}^{KM}(v_{k,K}^{(\alpha)}, v_{l,M}^{(\alpha)}) \prod_{N=1}^{\infty} \prod_{l'=1}^{N_{N|w}^{(\alpha)}} S_{vv}^{KN}(v_{k,K}^{(\alpha)}, w_{l',N}^{(\alpha)}) , \tag{3.11}$$

where the auxiliary S-matrix is given by (3.10).

In fact the coordinates $w_{l,N}^{(\alpha)}$ of the w-strings appearing in (3.11) can be excluded from the equation if one takes into account that the product of K equations in (3.8) with roots $w_k^{(\alpha)}$ that form a $K|vw$-string gives the following equation on the coordinates $v_{k,K}^{(\alpha)}$ of the $K|vw$-string

$$(-1)^K = \prod_{l=1}^{N_y^{(\alpha)}} \frac{v_{k,K}^{(\alpha)+} - v_l^{(\alpha)}}{v_{k,K}^{(\alpha)-} - v_l^{(\alpha)}} \prod_{N=1}^{\infty} \prod_{l=1}^{N_{N|w}^{(\alpha)}} S_{vv}^{KN}(v_{k,K}^{(\alpha)}, w_{l,N}^{(\alpha)}) . \tag{3.12}$$

Thus, (3.11) and (3.12) lead to the following equation

$$(-1)^K = \prod_{l=1}^{K^{\mathrm{I}}} S_{xv}^{Q_l K}(x_l, v_{k,K}^{(\alpha)}) \prod_{l=1}^{N_y^{(\alpha)}} \frac{v_{k,K}^{(\alpha)-} - v_l^{(\alpha)}}{v_{k,K}^{(\alpha)+} - v_l^{(\alpha)}} \prod_{M=1}^{\infty} \prod_{l=1}^{N_{M|vw}^{(\alpha)}} S_{vv}^{KM}(v_{k,K}^{(\alpha)}, v_{l,M}^{(\alpha)}) \tag{3.13}$$

which has no dependence on the coordinates of w-strings.

The set of the equations (3.1), (3.6), (3.11), (3.9) can be used to derive the TBA equations for the free energy of the mirror model. These equations and their consequences will be discussed in our forthcoming publication.

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A Mirror dispersion and parametrizations

The dispersion relation in any quantum field theory can be found by analyzing the pole structure of the corresponding two-point correlation function. Since the correlation function can be computed in Euclidean space, both dispersion relations in the original theory with H and in the mirror one with $\widetilde{H}$ are obtained from the following expression

$$H_{\rm E}^2 + 4g^2 \sin^2 \frac{p_{\rm E}}{2} + Q^2\,, \tag{A.1}$$

which appears in the pole of the 2-point correlation function. Here we consider the light-cone gauge-fixed string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ which has the Euclidean dispersion relation (A.1) for Q-particle bound states in the decompactification limit $L \equiv P_+ \to \infty$. The parameter g is the string tension, and is related to the 't Hooft coupling λ of the dual gauge theory as $g = \frac{\sqrt{\lambda}}{2\pi}$.

Then the dispersion relation in the original theory follows from the analytic continuation (see also [8])

$$H_{\rm E} \to -iH\,, \qquad p_{\rm E} \to p \quad \Rightarrow \quad H^2 = Q^2 + 4g^2 \sin^2 \frac{p}{2}\,, \tag{A.2}$$

and the mirror one from

$$H_{\rm E} \to \widetilde{p}\,, \qquad p_{\rm E} \to i\widetilde{H} \quad \Rightarrow \quad \widetilde{H} = 2\,\mathrm{arcsinh}\Big(\frac{1}{2g}\sqrt{Q^2 + \widetilde{p}^2}\Big)\,. \tag{A.3}$$

Comparing these formulae, we see that p and $\widetilde{p}$ are related by the following analytic continuation

$$p \to 2i\,\mathrm{arcsinh}\Big(\frac{1}{2g}\sqrt{Q^2 + \widetilde{p}^2}\Big)\,, \qquad H = \sqrt{Q^2 + 4g^2 \sin^2 \frac{p}{2}} \to i\widetilde{p}\,. \tag{A.4}$$

In what follows we need to know how the parameters $x^{Q\pm}$ which satisfy the relations

$$x^{Q+} + \frac{1}{x^{Q+}} - x^{Q-} - \frac{1}{x^{Q-}} = 2i\frac{Q}{g}\,, \quad \frac{x^{Q+}}{x^{Q-}} = e^{ip}\,, \tag{A.5}$$

are expressed through $\widetilde{p}$. By using formulae (A.4), we find

$$x^{Q\pm}(\widetilde{p}) = \frac{1}{2g}\left(\sqrt{1 + \frac{4g^2}{Q^2 + \widetilde{p}^2}} \mp 1\right)(\widetilde{p} - iQ)\,, \tag{A.6}$$

where we fix the sign of the square root from the conditions

$$\mathrm{Im}\,\big(x^{Q+}\big) < 0\,, \quad \mathrm{Im}\,\big(x^{Q-}\big) < 0 \;\; \text{for} \;\; \widetilde{p} \in \mathbf{R}\,. \tag{A.7}$$

As a consequence, one gets

$$ix^{Q-} - ix^{Q+} = \frac{i}{g}(\widetilde{p} - iQ) , \quad x^{Q+}x^{Q-} = \frac{\widetilde{p} - iQ}{\widetilde{p} + iQ} ,$$

which implies that $|x^{Q+}x^{Q-}| = 1$ and $|x^{Q+}| < |x^{Q-}|$ for $\widetilde{p}$ real. Also one has

$$x^{Q\pm}(-\tilde{p}) = -\frac{1}{x^{Q\mp}(\widetilde{p})} , \qquad (x^{Q\pm}(\widetilde{p}))^* = \frac{1}{x^{Q\mp}(\widetilde{p}^*)} \tag{A.8}$$

Note that these relations are well-defined for real $\widetilde{p}$, but one should use them with caution for complex values of $\widetilde{p}$. Our choice of the square root cut agrees with the one used in Mathematica: it goes over the negative semi-axes.

It what follows it will be often convenient to use the u-rapidity variables defined by

$$\begin{aligned} u &= \frac{1}{2}\left(x^{Q+} + \frac{1}{x^{Q+}} + x^{Q-} + \frac{1}{x^{Q-}}\right) = x^{Q+} + \frac{1}{x^{Q+}} - i\frac{Q}{g} = x^{Q-} + \frac{1}{x^{Q-}} + i\frac{Q}{g} , \\ u^{Q+} &= x^{Q+} + \frac{1}{x^{Q+}} = u + i\frac{Q}{g} , \quad u^{Q-} = x^{Q-} + \frac{1}{x^{Q-}} = u - i\frac{Q}{g} . \end{aligned} \tag{A.9}$$

The u-variable is expressed in terms of $\widetilde{p}$ as

$$u(\widetilde{p}) = \frac{\widetilde{p}}{g}\sqrt{1 + \frac{4g^2}{Q^2 + \widetilde{p}^2}} , \tag{A.10}$$

and it is an odd function of $\widetilde{p}$. The parameters $x^{Q\pm}$ and $\widetilde{p}$ are expressed in terms of u as follows

$$x^{Q+}(u) = \frac{1}{2}\left(u + \frac{iQ}{g} - i\sqrt{4 - \left(u + \frac{iQ}{g}\right)^2}\right) , \tag{A.11}$$

$$x^{Q-}(u) = \frac{1}{2}\left(u - \frac{iQ}{g} - i\sqrt{4 - \left(u - \frac{iQ}{g}\right)^2}\right) , \tag{A.12}$$

$$\widetilde{p}^Q(u) = \frac{ig}{2}\left(\sqrt{4 - \left(u + \frac{iQ}{g}\right)^2} - \sqrt{4 - \left(u - \frac{iQ}{g}\right)^2}\right) . \tag{A.13}$$

Here the cuts in the u-plane run from $\pm\infty$ to $\pm 2 \pm \frac{iQ}{g}$ along the horizontal lines. The u-plane with the cuts is mapped onto the region $\mathrm{Im}\left(x^{Q\pm}\right) < 0$ which is the physical region of the mirror theory, and therefore it is natural to expect that the u-plane should be used in all the considerations. To describe bound states for all values of $\widetilde{p}$ one should also add either the both lower or both upper edges of the cuts to the u-plane. They correspond $\mathrm{Im}\left(x^{Q\pm}\right) = 0$. This breaks the parity invariance of the model.

The energy of a Q-particle is expressed in terms of u as follows

$$\widetilde{\mathcal{E}}^Q(u) = \log\frac{x^{Q-}}{x^{Q+}} = 2\,\mathrm{arcsinh}\left(\frac{\sqrt{\left(u^2 + \sqrt{\frac{(u^2-4)^2g^4 + 2Q^2(u^2+4)g^2 + Q^4}{g^4}} - 4\right)g^2 + Q^2}}{2\sqrt{2}g}\right), \tag{A.14}$$

and it is positive for real values of u.

It would be also convenient to introduce the function

$$x(u) = \frac{1}{2}\left(u - i\sqrt{4-u^2}\right), \tag{A.15}$$

with the cuts in the u-plane run from $\pm\infty$ to ± 2 along the real lines, so that

$$x^{Q+}(u) = x\left(u + \frac{iQ}{g}\right), \quad x^{Q-}(u) = x\left(u - \frac{iQ}{g}\right). \tag{A.16}$$

Also one has

$$x^{Q\pm}(-u) = -\frac{1}{x^{Q\mp}(u)}, \; x(-u) = -\frac{1}{x(u)}, \quad (x^{Q\pm}(u))^* = \frac{1}{x^{Q\mp}(u^*)}, \; (x(u))^* = \frac{1}{x(u^*)},$$

and

$$\widetilde{p}(-u) = -\widetilde{p}(u), \qquad (\widetilde{p}(u))^* = \widetilde{p}(u^*). \tag{A.17}$$

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